Symmetry and dependence properties within a semiparametric family of bivariate copulas

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Abstract

In this paper, we study a semiparametric family of bivariate copulas. The family is generated by an univariate function, determining the symmetry (radial symmetry, joint symmetry) and dependence property (quadrant dependence, total positivity, ...) of the copulas. We provide bounds on different measures of association (such as Kendall's Tau, Spearman's Rho) for this family and several choices of generating functions allowing to reach these bounds.

Keywords: Copulas, semiparametric family, measures of association, positive dependence, dependence orderings.

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1 Introduction

The theory of copulas provides a relevant tool to build multivariate probability laws, from fixed margins and required degree of dependence. From Sklar's Theorem [18], the dependence properties of a continuous multivariate distribution H can be entirely summarized, independently of its margins, by a copula, uniquely associated with H. Several families of copulas, such as Archimedian copulas [4] or copulas with polynomial sections [15, 13] have been proposed. In [13], the authors point out that the copulas with quadratic section proposed in [15] are not able to modelize large dependences. Then, they introduce copulas with cubic sections and conclude that copulas with higher order polynomial sections would increase the dependence degrees but simultaneously the complexity of the model. In [5], polynomial-type extensions of the Farlie-Gumbel-Morgenstern bivariate distributions [2] are studied. It is shown that the positive correlation between the marginal distributions can be increased up to ≈ 0.39 but that the maximal negative correlation remains $-1/3$. Moreover, it has been remarked that dependence degrees arbitrarily close to ± 1 cannot be obtained with polynomial functions of fixed degree [6]. Thus, we propose to give up the polynomial form to work with a semiparametric family of copulas. The induced parametric families of copulas are generated as simply as Archimedian

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copulas, that is by an univariate function. Furthermore, dependence properties of copulas with polynomial sections are preserved and the dependence degree can be increased without significantly complexifying the model. Note that, in [3], a class of symmetric bivariate copulas with wide correlation coefficients range is introduced, but these copulas are less convenient to perform classical calculations on a probability law than the copulas studied here.

In Section 2, the semiparametric family is defined and its basic properties are derived. Symmetry properties are established in Section 3. Properties of three measures of association are investigated in Section 4. Section 5 is devoted to the dependence structure and the dependence ordering of the family.

2 Definition and basic properties

Throughout this paper, we note $I = [0, 1]$. A bivariate copula defined on the unit square I^2 is a bivariate cumulative distribution function with univariate uniform I margins. Equivalently, C must satisfy the following properties :

(P1) $C(u, 0) = C(0, v) = 0, \ \forall (u, v) \in I^2$,

(P2) $C(u, 1) = u$ and $C(1, v) = v, \ \forall (u, v) \in I^2$,

(P3) $\Delta(u_1, u_2, v_1, v_2) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0, \ \forall (u_1, u_2, v_1, v_2) \in I^4$, such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Let us recall that, from Sklar's Theorem, any bivariate distribution with cumulative distribution function H and marginal cumulative distribution functions F and G can be written $H(x, y) = C(F(x), G(y))$, where C is a copula. This result justifies the use of copulas for building bivariate distributions.

We consider the semiparametric family of functions defined on I^2 by :

$$C_\theta(u, v) = uv + \theta\phi(u)\phi(v), \ \ \theta \in [-1, 1], \tag{2.1}$$

where ϕ is a function on I. This family was first introduced in [16], chapter 3, and is a particular case of Farlie's family introduced in [2].

Let us note first that, the independent copula $C_0(u, v) = uv$ belongs to any parametric family $\{C_\theta\}$ generated by a function ϕ. Second, the functions ϕ and $(-\phi)$ clearly define the same function C_θ. The following theorem, very similar to Theorem 2.2 in [15], dedicated to bivariate copulas with quadratic sections, gives sufficient and necessary conditions on ϕ to ensure that C_θ is a copula.

Theorem 1 ϕ *generates a parametric family of copulas* $\{C_\theta, \ \theta \in [-1, 1]\}$ *if and only if it satisfies the following conditions :*

(a) $\phi(0) = \phi(1) = 0$,

(b) ϕ *satisfies the Lipschitz condition :* $|\phi(x) - \phi(y)| \leq |x - y|, \ \ \forall (x, y) \in I^2$.

Furthermore, C_θ is absolutely continuous.

Proof: It is clear that **(P1)**$\Leftrightarrow$ $(\phi(0) = 0)$ and **(P2)**$\Leftrightarrow$ $(\phi(1) = 0)$. In the case of the C_θ function, Δ rewrites

$$\Delta(u_1, u_2, v_1, v_2) = (u_2 - u_1)(v_2 - v_1) + \theta(\phi(u_2) - \phi(u_1))(\phi(v_2) - \phi(v_1)),$$

and then **(b)**$\Rightarrow$ **(P3)**. Conversely, if there are $x < y$ such that $|\phi(x) - \phi(y)| > |x - y|$, then with the choice $\theta = -1$, $u_1 = v_1 = x$ and $u_2 = v_2 = y$, we obtain:
$\Delta(x, y, x, y) = (x - y)^2 - (\phi(x) - \phi(y))^2 < 0$. Therefore **(P3)** $\Rightarrow$ **(b)**. Lastly, C_θ is absolutely continuous since ϕ satisfies the Lipschitz condition. $\qquad\square$

The following corollary provides a new characterization of the functions ϕ generating parametric families of copulas. It will be of great help in the following.

Corollary 1 *ϕ generates a parametric family of copulas $\{C_\theta,\ \theta \in [-1, 1]\}$ if and only if it satisfies the following conditions :*

(i) *ϕ est absolutely continuous,*

(ii) *$|\phi'(x)| \leq 1$ almost everywhere in I,*

(iii) *$|\phi(x)| \leq \min(x, 1 - x),\ \forall x \in I$.*

In such a case, C_θ is absolutely continuous.

Proof: From a classical analysis result ([19], Lemma 2.1), **(b)** is satisfied if and only if **(i)** and **(ii)** both hold. Now, assume that conditions **(i)**–**(iii)** are satisfied. Taking successively $x = 0$ and $x = 1$ in **(iii)** yields **(a)**. Therefore the corollary implies the theorem.
Conversely, assume that **(b)** holds. Taking successively $y = 0$ and $y = 1$ in the Lipschitz condition gives **(iii)**. So, the theorem implies the corollary. $\qquad\square$

The function ϕ plays a role similar to the generating function in Archimedian copulas [4]. Each copula C_θ is entirely described by the univariate function ϕ and the parameter θ, which tunes the dependence between the margins (see Section 4). Symmetry and dependence properties of the copula C_θ will found a geometrical interpretation on the graph of ϕ. Furthermore, the choice of ϕ determines the vertical and horizontal sections of the copula up to a multiplicative factor and an additive linear function.
We now give some examples of functions ϕ generating parametric families of copulas. These examples will be used to illustrate the symmetry and dependence properties introduced in the following sections.

Example 1 *The following functions ϕ generate parametric families of copulas.*

1. *$\phi^{[1]}(x) = \min(x, 1 - x)$ is the upper bound of Corollary 1**(iii)**.*

2. *$\phi^{[2]}(x) = x(1 - x)$ generates the Farlie-Gumbel-Morgenstern (FGM) family of copulas [10], which contains all copulas with both horizontal and vertical quadratic sections [15].*

3. *$\phi^{[3]}(x) = x(1 - x)(1 - 2x)$ defines the parametric family of symmetric copulas with cubic sections proposed in [13], equation (4.4).*

4. *$\phi^{[4]}(x) = \frac{1}{\pi} \sin(\pi x)$ induces a family of copulas able to modelize large dependences.*

In the sequel, we note $\{C_\theta^{[i]}\}$ the parametric family of copulas associated to the function $\phi^{[i]}$, $i \in \{1, \ldots, 4\}$. Graphes of the functions $\phi^{[i]}$ are plotted in [1]. As a consequence of the condition **(iii)** of Corollary 1, graphes of $\phi^{[1]}$ and $(-\phi^{[1]})$ are the edges of a square K, within which lies the graph of ϕ.

In the following, we review several concepts of symmetry and dependence. Throughout the rest of this paper, ϕ denotes a function satisfying the conditions given in Theorem 1, and $\{C_\theta\}$ represents the parametric family of copulas genetated by the function ϕ.

3 Symmetry properties

Let $(a, b) \in \mathbb{R}^2$ and (X, Y) a random pair. We say that X is symmetric about a if the cumulative distribution function of $(X - a)$ and $(a - X)$ are identical. The following definitions generalize this symmetry concept to the bivariate case:

- X and Y are exchangeable if (X, Y) and (Y, X) are identically distributed.

- (X, Y) is marginally symmetric about (a, b) if X and Y are symmetric about a and b respectively.

- (X, Y) is radially symmetric about (a, b) if $(X - a, Y - b)$ and $(a - X, b - Y)$ follow the same joint cumulative distribution function.

- (X, Y) is jointly symmetric about (a, b) if the pairs $(X-a, Y-b)$, $(a-X, b-Y)$, $(X-a, b-Y)$ and $(a - X, Y - b)$ have a common joint cumulative distribution function.

The following theorem provides conditions on ϕ to ensure that the couple (X, Y) with associated copula C_θ is radially (or jointly) symmetric.

Theorem 2

(i) *If X and Y are identically distributed then X and Y are exchangeable.*

Besides, if (X, Y) is marginally symmetric about (a, b) then:

(ii) *(X, Y) is radially symmetric about (a, b) if and only if*

$$\text{either } \forall u \in I, \ \phi(u) = \phi(1 - u) \text{ or } \forall u \in I, \ \phi(u) = -\phi(1 - u). \tag{3.1}$$

(iii) *(X, Y) is jointly symmetric about (a, b) if and only if $\forall u \in I, \ \phi(u) = -\phi(1 - u)$.*

Proof:

(i) When X and Y are identically distributed, exchangeability is equivalent to the symmetry of the copula. In other words, C_θ must verify $C_\theta(u, v) = C_\theta(v, u)$, $\forall (u, v) \in I^2$, which is the case by definition, see (2.1).

(ii) Assume (X, Y) is marginally symmetric about (a, b). Then, from Theorem 2.7.3 in [14], (X, Y) is radially symmetric about (a, b) if and only if

$$\forall (u, v) \in I^2, \ \delta_\theta(u, v) = 0, \tag{3.2}$$

where we have defined

$$\delta_\theta(u,v) = C_\theta(u,v) - C_\theta(1-u,1-v) - u - v + 1 = \theta[\phi(u)\phi(v) - \phi(1-u)\phi(1-v)].$$

It clearly appears that (3.1) implies (3.2). Conversely, suppose that (3.2) is verified. Then, $\delta_1(u,u) = \phi^2(u) - \phi^2(1-u) = 0$ for all $u \in I$ and consequently, either $\phi(u) = \phi(1-u)$ or $\phi(u) = -\phi(1-u)$. If there exist (u_1, u_2) such that $\phi(u_1) = \phi(1-u_1) \neq 0$ and $\phi(u_2) = -\phi(1-u_2) \neq 0$, then $\delta_1(u_1, u_2) = 2\phi(u_1)\phi(u_2) \neq 0$. As a conclusion (3.2) implies (3.1).

(iii) Assume (X,Y) is marginally symmetric about (a,b). (X,Y) is jointly symmetric about (a,b) if and only if

$$\forall (u,v) \in I^2, \quad \delta_\theta'(u,v) = 0, \tag{3.3}$$

where we have defined

$$\delta_\theta'(u,v) = C_\theta(u,v) + C_\theta(u,1-v) - u = \theta\phi(u)[\phi(v) + \phi(1-v)].$$

It appears immediately that (3.3) is equivalent to $\phi(u) = -\phi(1-u)$, $\forall u \in I$. $\qquad\square$

As an example, any marginally symmetric random pair (X,Y) associated to a copula of the families $\{C_\theta^{[i]}\}$, $i \in \{1, \ldots, 4\}$ is radially symmetric. Moreover, in the case $i = 3$, (X,Y) is jointly symmetric. We now focus on the dependence properties of the semiparametric family of copulas.

4   Measures of association

In the next two sections, we note (X,Y) a random pair with joint distribution H, density distribution h, copula C and margins F and G. The case $C = C_\theta$ will be explicitly precised. Three invariant to strictly increasing function measures of association between the components of the random pair (X,Y) are usually considered:

- the normalized volume between graphes of H and FG [17],

$$\sigma = 12 \int_0^1 \int_0^1 |C(u,v) - uv| \, du \, dv,$$

- the Kendall's Tau [2,3], defined as the probability of concordance minus the probability of discordance of two pairs (X_1, Y_1) and (X_2, Y_2) described by the same joint bivariate law H,

$$\tau = 4 \int_0^1 \int_0^1 C(u,v) \, dC(u,v) - 1, \tag{4.1}$$

- the Spearman's Rho [2,3], which is the probability of concordance minus the probability of discordance of two pairs (X_1, Y_1) and (X_2, Y_2) with respective joint cumulative law H and FG,

$$\rho = 12 \int_0^1 \int_0^1 C(u,v) \, du \, dv - 3.$$

In the case of a copula generated by (2.1), these measures rewrite only in terms of the function ϕ.

Proposition 1 *Let (X, Y) be a random pair with copula C_θ given by (2.1). The association coefficients are:*

$$\sigma_\theta = 12|\theta| \left(\int_0^1 |\phi(u)|du \right)^2, \quad \tau_\theta = 8\theta \left(\int_0^1 \phi(u)du \right)^2, \quad \rho_\theta = 12\theta \left(\int_0^1 \phi(u)du \right)^2 = \frac{3}{2}\tau_\theta.$$

Proof: The proof are very similar for the three coefficients. Let us consider the example of the Kendall's Tau. The density distribution of the cumulative distribution function C_θ is

$$c_\theta(u, v) = 1 + \theta\phi'(u)\phi'(v). \tag{4.2}$$

Replacing in (4.1), it yields

$$\begin{aligned}
\tau_\theta &= 4\theta \left[\int_0^1 \phi(u)du \right]^2 + 4\theta \left[\int_0^1 u\phi'(u)du \right]^2 + 4\theta^2 \left[\int_0^1 \phi(u)\phi'(u)du \right]^2, \\
&= 4\theta \left[\int_0^1 \phi(u)du \right]^2 + 4\theta \left[\phi(1) - \int_0^1 \phi(u)du \right]^2 + 2\theta^2[\phi^2(1) - \phi^2(0)],
\end{aligned}$$

after partial integration. Theorem 1 entails $\phi(1) = \phi(0) = 0$ and the conclusion follows. □

The measures τ_θ and ρ_θ linearly increase with θ, which appears as an association parameter. They are also proportionnals to the square of the surface lying between the graph of ϕ and the x-axis. Bounds for each of these measures of association are deduced from Corollary 1 **(iii)**.

Proposition 2 $\forall \theta \in [-1, 1]$, $0 \le \sigma_\theta \le 3|\theta|/4$, $|\tau_\theta| \le |\theta|/2$ and $|\rho_\theta| \le 3|\theta|/4$.

Therefore the range of each association coefficient is $0 \le \sigma_\theta \le 3/4$, $-1/2 \le \tau_\theta \le 1/2$ and $-3/4 \le \rho_\theta \le 3/4$. The semiparametric family of copulas defined by (2.1) seems to be a good tool to build low complexity models with moderate dependences ($0 \le |\tau| \le 1/2$). This is illustrated in Table 1 where the values obtained for the copulas presented in Example 1 are reported. Results obtained for the cubic sections copulas illustrate the fact that τ and ρ imperfectly measure the association relationships. All the values associated to $\{C_\theta^{[4]}\}$ are larger than those obtained with the FGM family. Lastly, bounds of Proposition 2 are reached with family $\{C_\theta^{[1]}\}$. This hierarchy of results graphically appears in [1]. For each fixed θ, the closer to the edges of the square K is the graph of ϕ, the larger is the association coefficient of the copula induced by ϕ. Non-derivability of $\phi^{[1]}$ can be a drawback. In order to build more regular copulas, with measures of association remaining closed from optimal values, we define the sequence of $\mathcal{C}^1$ functions $(\phi_n^{[5]})$ as following:

$$\forall n > 0, \ \phi_n^{[5]}(x) = \left| \begin{array}{ll} x & \text{if} \quad 0 \le x \le \dfrac{1}{2} - \dfrac{1}{n} \\[2mm] \dfrac{1}{2}\left(1 - \dfrac{1}{n}\right) - \dfrac{n}{2}\left(x - \dfrac{1}{2}\right)^2 & \text{if} \quad \dfrac{1}{2} - \dfrac{1}{n} < x < \dfrac{1}{2} + \dfrac{1}{n} \\[2mm] 1 - x & \text{if} \quad \dfrac{1}{2} + \dfrac{1}{n} \le x \le 1. \end{array} \right.$$

Moreover each copula is derivable and the associated Kendall's Tau sequence is given by

$$\tau_{n,\theta}^{[5]} = 8\theta \left(\frac{1}{4} - \frac{1}{3n^2} \right)^2 , \quad n > 0.$$

It converges to the optimal Tau, $\tau_{n,\theta}^{[5]} \to \theta/2$ as $n \to \infty$. It is also possible to build $\mathcal{C}^\infty$ copulas, with measures of association closed from optimal values. Define the sequence $(\phi_n^{[6]})$ of $\mathcal{C}^\infty$ functions by:

$$\forall n \geq 2, \quad \phi_n^{[6]}(x) = 1 - (x^n + (1-x)^n)^{1/n} .$$

Similarly to the previous example, for each $n \geq 2$, $\phi_n^{[6]}$ induces a parametric family of copulas $\{C_{n,\theta}^{[6]}\}$. Moreover $\forall x \in I$, $\phi_n^{[6]}(x) \to \phi^{[1]}(x)$ as $n \to \infty$ and $|\phi_n^{[6]}| \leq 1$. So, by Lebesgue dominated convergence theorem, the associated Kendall's Tau, $\tau_{n,\theta}^{[6]}$ converges to the optimal value as $n \to \infty$. Graphes of $\phi_2^{[6]}$, $\phi_4^{[6]}$, and $\phi_8^{[6]}$ are plotted in [1]. We clearly see that the surface lying between the $\phi_n^{[6]}$ graph and the x-axis increases with n. The sequences $(\tau_{n,1}^{[6]})$ and $(\tau_{n,1}^{[5]})$ are plotted in [1]. We can observe that for $n \geq 2$, $\tau_{n,1}^{[5]}$ and $\tau_{n,1}^{[6]}$ are both larger than the value obtained by the FGM copula $C_1^{[2]}$ and that they exceed the value reached by the copula $C_1^{[4]}$, whenever $n \geq 3$.

5 Concepts of dependence

Although the notion of independence between X and Y is clearly defined by $H = FG$, the concept of dependence may be defined in different ways. A dependence ordering, indicating if the random pair (X_1, Y_1) of margins F and G is "more dependent" than the pair (X_2, Y_2) with the same margins, can be associated to each definition. We firstly study the properties of positive dependence of a copula C_θ and secondly we search to order elements within the parametric family $\{C_\theta\}$.

5.1 Positive dependence

In this subsection, we will use several concepts of positive dependence, which express that two variables are large (or small) simultaneously. Let us shortly review their definition.

- PFD: X and Y are Positive Function Dependent if for all integrable real-valued function g $\mathbb{E}_h[g(X)g(Y)] - \mathbb{E}_h[g(X)]\mathbb{E}_h[g(Y)] \geq 0$ where $\mathbb{E}_h$ is the expectation symbol relative to the density h.

- PQD: X and Y are Positively Quadrant Dependent if

$$\forall (x,y) \in \mathbb{R}^2, \quad P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y). \tag{5.1}$$

- LTD$(Y|X)$: Y is Left Tail Decreasing in X if $P(Y \leq y|X \leq x)$ is nonincreasing in x for all y. A similar definition can be given for $LTD(X|Y)$.

- RTI$(Y|X)$: Y is Right Tail Increasing in X if $P(Y > y|X > x)$ is nondecreasing in x for all y.

- SI($Y|X$): Y is Stochastically Increasing in X if $P(Y > y|X = x)$ is nondecreasing in x for all y.

- LCSD: X and Y are Left Corner Set Decreasing if $P(X \leq x, Y \leq y|X \leq x', Y \leq y')$ is nonincreasing in x' and y' for all x and y.

- RCSI: X and Y are Right Corner Set Increasing if $P(X > x, Y > y|X > x', Y > y')$ is nondecreasing in x' and y' for all x and y.

- TP2 density: (X, Y) have the TP2 density property if h is a totally positive function of order 2 i.e. $h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1) \geq 0$ for all $(x_1, x_2, y_1, y_2) \in I^4$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

When X and Y are exchangeable, there are no reason to distinguish SI($Y|X$) and SI($X|Y$), which will be both noted SI. Similarly, we will denote LTD the equivalent properties LTD($Y|X$) and LTD($X|Y$), and RTI, RTI($Y|X$) or RTI($X|Y$). The links between these concepts [9, 11] are illustrated in [1]. The following theorem is devoted to the study of properties of positive dependence of any pair (X, Y) associated with the copula C_θ defined by (2.1). Similar results can be established for the corresponding concepts of negative dependence.

Theorem 3 *Let $\theta > 0$ and (X, Y) a random pair with copula C_θ.*

(i) *X and Y are PFD.*

(ii) *X and Y are PQD if and only if either $\forall u \in I$, $\phi(u) \geq 0$ or $\forall u \in I$, $\phi(u) \leq 0$.*

(iii) *X and Y are LTD if and only if $\phi(u)/u$ is monotone.*

(iv) *X and Y are RTI if and only if $\phi(u)/(u-1)$ is monotone.*

(v) *X and Y are LCSD if and only if they are LTD.*

(vi) *X and Y are RCSI if and only if they are RTI.*

(vii) *X and Y are SI if and only if ϕ is either concave or convex.*

(viii) *X and Y have the TP2 density property if and only if they are SI.*

Proof:

(i) Let g be an integrable real-valued function on I. The density distribution c_θ of the cumulative distribution C_θ is given by (4.2). Routine calculations yield

$$\mathbb{E}_{c_\theta}(g(X)g(Y)) - \mathbb{E}_{c_\theta}(g(X))\mathbb{E}_{c_\theta}(g(Y)) = \theta \left[\int_0^1 g(t)\phi'(t)dt \right]^2 \geq 0,$$

since $\theta \geq 0$.

(ii) The pair (X, Y) is PQD if and only if the uniform I-margins pair (U, V) with distribution C_θ is PQD. For (U, V), condition (5.1) simply rewrites $\theta\phi(u)\phi(v) \geq 0$, $\forall (u, v) \in I^2$ and the conclusion follows.

(iii) Since C_θ is symmetric, the necessary and sufficient conditions given by Theorem 5.2.5 in [14], simply reduce to the unique condition: $C_\theta(u,v)/u = v + \theta\phi(v)\phi(u)/u$ is nonincreasing in u for all $v \in I$. Suppose for instance $\phi(u)/u$ is nonincreasing. Then, since $\phi(1) = 0$, we have $\phi(v) \geq 0$ for all v in I and thus $C_\theta(u,v)/u$ is nonincreasing in u for all $v \in I$. The case of a nondecreasing function $\phi(u)/u$ is similar.

Conversely, suppose $C_\theta(u,v)/u$ is nonincreasing in u for all $v \in I$. If there exist $v \in I$ such that $\phi(v) > 0$ then $\phi(u)/u$ is nonincreasing too. The case where there is $v \in I$ such that $\phi(v) < 0$ is similar. The case $\phi(v) = 0$ for all $v \in I$ is trivial. As a conclusion, we have shown that $C_\theta(u,v)/u$ is nonincreasing in u for all $v \in I$ if and only if $\phi(u)/u$ is monotone.

(iv) is similar to **(iii)**.

(v) In view of Corollary 5.2.17 in [14], a necessary and sufficient condition for X and Y LCSD is: C_θ is a totally positive function of order 2. For all $u_1 \leq u_2$ and $v_1 \leq v_2$, the quantity

$$C_\theta(u_1,v_1)C_\theta(u_2,v_2) - C_\theta(u_1,v_2)C_\theta(u_2,v_1) = \theta\left[\frac{\phi(v_2)}{v_2} - \frac{\phi(v_1)}{v_1}\right]\left[\frac{\phi(u_2)}{u_2} - \frac{\phi(u_1)}{u_1}\right]u_1u_2v_1v_2$$

is nonnegative if and only if $\phi(u)/u$ is monotone. This is the necessary and sufficient condition for X and Y LTD given in **(iii)**.

(vi) is similar to **(v)**.

(vii) In view of the C_θ symmetry, the geometric interpretation of stochastic monotonicity given by Corollary 5.2.11 in [14] provides the necessary and sufficient condition: $C_\theta(u,v)$ is a concave function of u. Suppose ϕ is a concave function. Then, taking into account that $\phi(0) = \phi(1) = 1$, we easily show that $\phi(v) \geq 0$ for all $v \in I$ and therefore $C_\theta(u,v)$ is a concave function of u. The case of a convex function ϕ is similar.

Conversely, suppose $C_\theta(u,v)$ is a concave function of u. If there exist v such that $\phi(v) > 0$ then (2.1) shows that ϕ is concave. The case where there exist $v \in I$ such that $\phi(v) < 0$ is similar. The case $\phi(v) = 0$ for all $v \in I$ is trivial. As a conclusion we have shown the equivalence between $C_\theta(u,v)$ is a concave function of u and ϕ is concave or convex.

(viii) X and Y have the TP2 density property if and only if the density of the copula verifies

$$\forall u_1 \leq u_2,\ v_1 \leq v_2,\ \ c_\theta(u_1,v_1)c_\theta(u_2,v_2) - c_\theta(u_1,v_2)c_\theta(u_2,v_1) \geq 0,$$

which rewrites

$$\forall u_1 \leq u_2,\ v_1 \leq v_2,\ \ [\phi'(u_1) - \phi'(u_2)][\phi'(v_1) - \phi'(v_2)] \geq 0.$$

Equivalently, ϕ' is either nonincreasing or nondecreasing which means that ϕ is either concave or convex. This is the necessary and sufficient condition for X and Y SI established in **(vii)**. $\square$

Results of Theorem 3 are illustrated in [1]. For instance, random pairs with copula in the parametric families generated by $\phi^{[1]}$, $\phi^{[2]}$, $\phi^{[4]}$, $\phi_n^{[5]}$ or $\phi_n^{[6]}$ have the TP2 density property since these functions are concave. Consequently, these random pairs are also SI, LCSD, RCSI, LTD, RTI, PQD, and PFD. When the copula belongs to the family $\{C_\theta^{[3]}\}$, the associated random pair is PFD but not PQD.

Finally, let us note that Theorem 3 provides a very simple tool to prove that the family proposed in [8]: $c(u,v) = uv + \theta u^b v^b(1-u)^a(1-v)^a$, $\theta \in [-1,1]$, $a \geq 1$, $b \geq 1$ is PQD.

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5.2 Dependence orderings

In Theorem 3 positive dependence properties of the copula C_θ are established for fixed $\theta > 0$. In this subsection we investigate the variations of the "dependence amount" when θ increases. To this end, it is necessary to define a dependence ordering which can decide whether a multivariate cumulative distribution function is more dependent than another. Consider H and H^* two bivariate cumulative distribution functions with the same margins. We study two dependence orderings:

- H^* is more PQD (or more concordant) than H if $H(x_1, x_2) \leq H^*(x_1, x_2)$ for all (x_1, x_2). We note $H \prec_C H^*$.

- Note $H_{2|1}$ and $H^*{}_{2|1}$ the conditional distributions of the second random variable given the first one. H^* is more SI than H if $H^*{}_{2|1}^{-1}(H_{2|1}(x_2|x_1)|x_1)$ is increasing in x_1. We note $H \prec_{SI} H^*$.

Remark that $H \prec_{SI} H^*$ implies $H \prec_C H^*$ (see Theorem 2.12 in [7]). The natural extension of these definitions to parametric families of copulas is the following.

- A family of copulas $\{C_\theta\}$ is ordered in concordance if $\theta \leq \theta'$ implies $C_\theta \prec_C C_{\theta'}$.

- A family of copulas $\{C_\theta\}$ is SI ordered if $\theta \leq \theta'$ implies $C_\theta \prec_{SI} C_{\theta'}$.

As mentionned in [7], $C_\theta \prec_C C_{\theta'}$ implies $\tau_\theta \leq \tau_{\theta'}$ and $\rho_\theta \leq \rho_{\theta'}$. We now give conditions under which the parametric family of copulas obtained by choosing ϕ according to Corollary 1 is stochasticly ordered.

Theorem 4

(i) $\{C_\theta\}$ *is ordered in concordance if and only if ϕ is either positive or negative.*

(ii) *Assume ϕ is 2-derivable. $\{C_\theta\}$ is SI ordered if ϕ is either convex or concave.*

Proof:

(i) The proof is straigtforward.

(ii) A direct proof is not possible since, in the general case, the inverse of the conditional cumulative distribution function does not have a closed-form. A necessary condition is given by Theorem 2.14 in [7]: $\{C_\theta\}$ is SI ordered if for all increasing real-valued function g,

$$A(u, v, \theta) = \frac{\partial^2 B}{\partial v \partial u} \frac{\partial B}{\partial \theta} - \frac{\partial^2 B}{\partial \theta \partial u} \frac{\partial B}{\partial v} \geq 0,$$

where $B(u, v, \theta) = g(v + \theta \phi'(u)\phi(v))$. We obtain

$$A(u, v, \theta) = -\phi''(u)\phi(v)[g'(v + \theta \phi'(u)\phi(v))]^2.$$

The sign of $A(u, v, \theta)$ is given by $-\phi''(u)\phi(v)$, which is positive whenever ϕ is concave or convex. $\qquad \square$

For instance, all families discussed in this document are ordered in concordance, but $\{C_\theta^{[3]}\}$ is not SI ordered.

6   Conclusion and further work

In this paper, a symmetric semiparametric family of copulas is studied and its symmetry and dependence properties are established. As for Archimedian copulas, a parametric subfamily is generated by an univariate function and both symmetry and dependence properties of the family can be geometrically interpreted on the graph of the generating function. Furthermore, the horizontal and vertical sections of the copula are essentially described by the generating function. Finally, the proposed family of copulas provides a "simple" way to modelize moderate degrees of dependence, in the sense that the copulas are convenient to perform classical calculations on a probability law. Our further work will consist in generalizing the family definition to the asymmetric case:

$$C_\theta(u,v) = uv + \theta a(u)b(v), \ \theta \in [-1,1], \tag{6.1}$$

where a and b are functions defined on I. This family was introduced in [20], chapter 4. It can be shown that, similarly to Theorem 1, necessary and sufficient conditions for (6.1) to be a copula are $a(0) = a(1) = b(0) = b(1) = 0$ and $|a(x) - a(y)||b(z) - b(t)| \le |x - y||z - t|$ for all $(x, y, z, t) \in I^4$. The study of the related measures of association and dependence concepts will be the topic of a next paper.

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$\phi(x)$	σ_θ	τ_θ	ρ_θ
$\phi^{[1]}(x) = \min(x, 1-x)$	$3\lvert\theta\rvert/4$	$\theta/2$	$3\theta/4$
$\phi^{[2]}(x) = x(1-x)$	$\lvert\theta\rvert/3$	$2\theta/9$	$\theta/3$
$\phi^{[3]}(x) = x(1-x)(1-2x)$	$3\lvert\theta\rvert/64$	0	0
$\phi^{[4]}(x) = \frac{1}{\pi}\sin(\pi x)$	$48\lvert\theta\rvert/\pi^4$	$32\theta/\pi^4$	$48\theta/\pi^4$

Table 1: Measures of dependence associated to the Example 1 copulas.